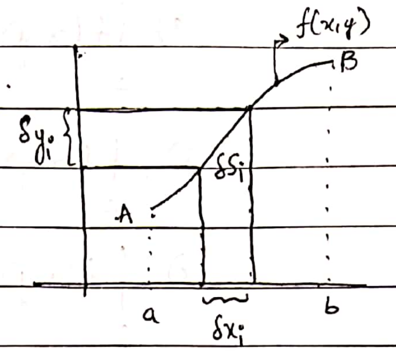
LINE AND MULTIPLE INTEGRALS

Def^n: A line integral is an integral of a function defined along a curve. Hence, integral of bounded functions defined over curves is called line integral. The curves may be plane curves or space curves.

LINE INTEGRAL OVER A PLANE CURVE:



Let $x=\phi(t)$, $y=\psi(t)$, $a\le t\le b$ be a curve C.
Consider, a partition of $[a,b]$. (ξ_i,η_i) be an arbitrary point of the segment δs_i whose projections on the axes are δx_i and δy_i. Then

$$\sum_{i=1}^{n} f(\xi_i,\eta_i)\,\delta x_i \;;\; \sum_{i=1}^{n} f(\xi_i,\eta_i)\,\delta y_i \;;\; \sum_{i=1}^{n} f(\xi_i,\eta_i)\,\delta s_i$$

As $n\to\infty$, the quantities $\delta x_i, \delta y_i, \delta s_i$ approaches to 0.
Then, $\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$, $\int_C f(x,y)\,ds$ are called line integral of the function $f(x,y)$ along C.

PROPERTIES

1. $\int_C K f(x,y)\,dx = K\int_C f(x,y)\,dx$, K = constant.

2. $\int_C (f\pm g)\,dx = \int_C f\,dx \pm \int_C g\,dx$.

3. $\int_A^B f(x,y)\,dx = -\int_B^A f(x,y)\,dx$

4. $\int_A^B f(x,y)\,dx + g(x,y)\,dy = \int_A^B f(x,y)\,dx + \int_A^B g(x,y)\,dy$

5. $\int_{AB} f(x,y)\,dx = \int_{AC} f(x,y)\,dx + \int_{CB} f(x,y)\,dx$

NOTE

1. If the equation of the curve C in the xy plane is $y=\phi(x)$. Then,

a) $\int_C f(x,y)\,dx = \int_{x_1}^{x_2} f(x,\phi(x))\,dx$

b) $\int_C f(x,y)\,dy = \int_{x_1}^{x_2} f(x,\phi(x))\,\phi'(x)\,dx$

c) $\int_C f(x,y)\,ds = \int_{x_1}^{x_2} f(x,\phi(x))\sqrt{1+(f'(x))^2}\,dx$

2) If the equation of the Curve is in the Parametric form $x=\phi(t)$, $y=\psi(t)$. Then

a) $\int_C f(x,y)\,dx = \int_{t_1}^{t_2} f[\phi(t),\psi(t)]\,\phi'(t)\,dt$

b) $\int_C f(x,y)\,dy = \int_{t_1}^{t_2} f[\phi(t),\psi(t)]\,\psi'(t)\,dt$

c) $\int_C f(x,y)\,ds = \int_{t_1}^{t_2} f[\phi(t),\psi(t)]\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

LINE INTEGRAL OVER A SPACE CURVE

Let $x=\phi_1(t)$, $y=\phi_2(t)$, $z=\phi_3(t)$, $a\le t\le b$ be the Curve G joining the points A and B. Divide G between A and B and δs_i whose ~~the~~ projections on the axes are δx_i, δy_i, δz_i. Let $(\xi_i,\eta_i,\varsigma_i)$ be a point in each δs_i.

The sums $\sum_{i=1}^{n} f(\xi_i,\eta_i,\varsigma_i)\,\delta x_i$, $\sum_{i=1}^{n} f(\xi_i,\eta_i,\varsigma_i)\,\delta y_i$,

$\sum_{i=1}^{n} f(\xi_i,\eta_i,\varsigma_i)\,\delta z_i$, $\sum_{i=1}^{n} f(\xi_i,\eta_i,\varsigma_i)\,\delta s_i$

As $n\to\infty$. The integrals will be $\int_C f(x,y,z)\,dx$,

$\int_C f(x,y,z)\,dy$, $\int_C f(x,y,z)\,dz$ and $\int_C f(x,y,z)\,ds$

Problems:

1. Evaluate $\int_C (5xy\,dx+y^2\,dy)$ 'C' is the Curve in the xy plane $y=2x^2$ from $(0,0)$ to $(1,2)$

Soln

$y=2x^2$

$dy=4x\,dx$, $0\le x\le 1$

$\int_C 5xy\,dx+y^2\,dy = \int_0^1 5x(2x^2)\,dx+\int_0^1 4x^4\times 4x\,dx$

$=\int_0^1 (10x^3+16x^5)\,dx = \left.10\frac{x^4}{4}+16\frac{x^6}{6}\right|_0^1$

$=\frac{10}{4}+\frac{16}{6}=\frac{31}{6}$

② $\int_C [(2x+y)dx + (3y+x)dy]$ along (0,1) & (2,5)

Sol^n Line joining (0,1) & (2,5) is

$\frac{y-1}{4} = \frac{x}{2} \Rightarrow y = 2x+1$

$\therefore dy = 2dx$.

x varies from 0 to 2

$\therefore \int_C (2x+y)dx + (3y+x)dy = \int_0^2 (2x+2x+1)dx + (3(2x+1)+x)2dx$

$= \int_0^2 (4x+1+12x+6+2x)dx = \int_0^2 (18x+7)dx$

$= \left(\frac{18}{2}x^2 + 7x\right)_0^2 = 36+14 = 50$

③ Evaluate $\int_C (x+y)dx + (y-x)dy$

a) along $y^2 = x$ from (1,1) to (4,2)

b) along $x = 2t^2+t+1$, $y = t^2+1$ & $0 \le t \le 1$

Sol^n a) $x = y^2$

$dx = 2y\,dy$ $\quad$ y varies 1 to 2

$\int_1^2 [(x+y)dx + (y-x)dy] = \int_1^2 (y^2+y)2y\,dy + (y-y^2)dy$

$= \int_1^2 (2y^3+2y^2+y-y^2)dy = \int_1^2 (2y^3+y^2+y)dy$

$= \left[\frac{2y^4}{4} + \frac{y^3}{3} + \frac{y^2}{2}\right]_1^2 = \left(\frac{16}{2} + \frac{8}{3} + 2\right) - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2}\right)$

$= 34/3$

b) $x = 2t^2+t+1$, $y = t^2+1$

$dx = (4t+1)dt$, $dy = 2t\,dt$

$\int_0^1 (x+y)dx + (y-x)dy = \int_0^1 (2t^2+t+1+t^2+1) + (4t+1)dt$

$\int_0^1 (t^2+1-(2t^2+t+1))2t\,dt$

$= \int_0^1 \left[(3t^2+t+2)(4t+1) + (-t^2-t)2t\right]dt$

$= \int_0^1 (10t^3+5t^2+9t+2)dt$

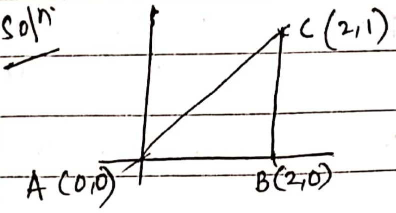
$= \left[\frac{10t^4}{4} + \frac{5t^3}{3} + \frac{9t^2}{2} + 2t\right]_0^1$

$= \frac{5}{2} + \frac{5}{3} + \frac{9}{2} + 2 = \frac{64}{6} = \frac{32}{3}$

④ $\int_C (2x^2+y^2)dx + (3y-4x)dy$

along the Δ^{le} ABC whose Vertices are A=(0,0), B=(2,0), C=(2,1)

Sol^n



Along AB:

$y = 0$

$dy = 0$, x varies 0 to 2

$\therefore \int_0^2 (2x^2+y^2)dx + (3y-4x)dy$

$= \int_0^2 2x^2 dx = 16/3$

Along BC:

$x = 2$

$dx = 0$; y varies from 0 to 1

$\int_0^1 (3y-8)dy = \frac{3}{2} - 8 = -\frac{13}{2}$

Along CA: $x = 2y$, $dx = 2dy$

$\int_1^0 (18y^2-5y)dy = \left(18\frac{y^3}{3} - 5\frac{y^2}{2}\right)_1^0$

$= 7/2$

$\therefore \int_C (2x^2+y^2)dx + (3y-4x)dx$

$= \frac{16}{3} - \frac{13}{2} - \frac{7}{2} = -\frac{14}{3}$

⑥ $\int_c [(x+2y)dx+(4-2x)dy]$ around $x^2/9+y^2/4=1$

Sol^n Parametric eq^n's of ellipse are $x=3\cos\theta,\ y=2\sin\theta$

$dx=-3\sin\theta\, d\theta,\ dy=2\cos\theta\, d\theta,$

$0\le\theta\le 2\pi$

$$\int_c (x+2y)dx+(4-2x)dy = \int_0^{2\pi}(3\cos\theta+4\sin\theta)(-3\sin\theta)d\theta + (4-6\cos\theta)2\cos\theta\, d\theta$$

$$=\int_0^{2\pi}(-9\cos\theta\sin\theta-12\sin^2\theta+8\cos\theta-12\cos^2\theta)d\theta$$

$$=\int_0^{2\pi}(8\cos\theta-12-9\cos\theta\sin\theta)d\theta$$

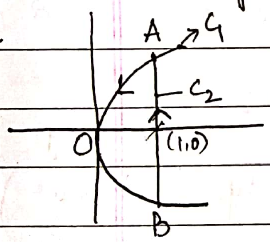
$$=\int_0^{2\pi}\left(8\cos\theta-\frac{9}{2}\sin2\theta-12\right)d\theta$$

$$=\left[8\sin\theta+\frac{9}{2}\frac{\cos2\theta}{2}-12\theta\right]_0^{2\pi}$$

$$=\left(8\sin2\pi+\frac{9}{4}\cos2\cdot2\pi-12\ 2\pi\right)-\left(0+9/4\right) = -24\pi$$

⑦ $\int_c [(x^2+2xy^2)dx+(x^2y^2-1)dy]$ around $y^2=4x$ and $x=1$

Sol^n



$A=(1,2)$
$B=(1,-2)$
$C: C_1+C_2$.

Along C_1 : $y^2=4x \Rightarrow y=2\sqrt{x}$

$dy=\frac{1}{\sqrt{x}}dx$

Also x varies b/w 0 & 1

$$\int_{C_1}[x^2+2xy^2]dx+(x^2y^2-1)dy] = \int_0^1 (x^2+2x.4x)dx+(x^2.4x-1)\frac{1}{\sqrt{x}}dx$$

$$=\int_0^1\left(9x^2+4x^{5/2}-x^{-1/2}\right)dx$$

$$=\left[9\frac{x^3}{3}+4\frac{x^{7/2}}{7/2}-\frac{x^{1/2}}{1/2}\right]_0^1$$

$= 15/7$.

Along C_2 : $x=1$; y varies from -2 to 2.
$dx=0$

$$\int_{C_2}\left[(x^2+2xy^2)dx+(x^2y^2-1)dy\right]$$

$$=\int_{-2}^{2}(y^2-1)dy=\left(\frac{y^3}{3}-y\right)_{-2}^{2}=\frac{4}{3}$$

$\therefore \int_c f(x,y)=\frac{15}{7}+\frac{4}{3}=\frac{73}{21}$ //.

⑧ $\int_c (2x+y)\,ds$ around $x^2+y^2=1$

Sol^n Parametric eq^n of circles are $x=\cos\theta,\ y=\sin\theta$. If 's' is the arc length then,

$$\left(\frac{ds}{d\theta}\right)^2=\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\sin^2\theta+\cos^2\theta=1$$

$\Rightarrow \frac{ds}{d\theta}=1$

The parameter 'θ' varies from 0 to 2π.

$$\int_c (2x+y)\frac{ds}{d\theta}d\theta=\int_0^{2\pi}(2\cos\theta+\sin\theta)d\theta$$

$$=(2\sin\theta-\cos\theta)_0^{2\pi}$$

$$=(2\sin2\pi-\cos2\pi)-(0-1)$$

$$=0$$

(9) $\int_c (x+y+z)\,ds$ 'c' is the line joining $(1,2,3)$ & $(4,5,6)$

Soln
Eqn of the line joining $(1,2,3)$ & $(4,5,6)$

$$\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3} = t$$

$\Rightarrow x = 3t+1,\ y = 3t+2,\ z = 3t+3$

$\Rightarrow \frac{dx}{dt} = 3,\ \frac{dy}{dt} = 3,\ \frac{dz}{dt} = 3$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= 27$$

$$\frac{ds}{dt} = 3\sqrt{3} \Rightarrow ds = 3\sqrt{3}\,dt$$

When $x=1, y=2, z=3$ we have $t=0$
When $x=4, y=5, z=6$ " $t=3$

$$\int_c (x+y+z)\frac{ds}{dt}\,dt = \int_0^3 (3t+1+3t+2+3t+3)3\sqrt{3}\,dt$$

$$= \int_0^3 (9t+6)3\sqrt{3}\,dt = 3\sqrt{3}\left\{9\frac{t^2}{2}+6t\right\}_0^3 = \frac{351\sqrt{3}}{2}$$

Problems

1) Evaluate $\int_c (xy\,dx + yz\,dy + zx\,dz)$ where c is $x=t,\ y=t^2,\ z=t^3$, t varying from -1 to 1.

2) Evaluate $\int_c (x^2+2y)\,dx + (3x-y)\,dy$ along the curve $x=2t,\ y=t^2+3$, $0 \le t \le 1$

3) Evaluate $\int_c (3x+y)\,dx + (2y-x)\,dy$ along $y = x^2+1$ from $(0,1)$ to $(3,10)$

4) Evaluate $\int_c \left[(x^2-2xy)\,dx + (x^2y+3)\,dy\right]$ around the region defined by $y^2 = 8x$ and $x=2$

5) $\int_c xy\,dx + x^2z\,dy + xyz\,dz$ where c is $x=e^t,\ y=e^{-t},\ z=t^2$ and $0 \le t \le 1$.

6) Evaluate $\int_c x^2y^2\,ds$ around the circle $x^2+y^2=1$

7) Evaluate $\int_{(0,1)}^{(2,3)} (2xy-1)\,dx + (x^2+1)\,dy$ along
a) line $y = x+1$
b) curve $y = \frac{x^2}{2}+1$

8) Evaluate $\int_c 3x^2\,dx + (2xz-y)\,dy + z\,dz$ along the line joining $(0,0,0)$ & $(2,1,3)$.

9) Evaluate the line integral of $\frac{a^2y^2}{b^2} + \frac{b^2x^2}{a^2}$ around the ellipse C whose eqn is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

10) Evaluate $\int_{(0,0)}^{(1,1)} x\,ds$ along i) $y=x$ ii) $y=x^2$

11) $\int_c x^2y\,dx + (x-z)\,dy + xyz\,dz$ where 'c' is the arc of the parabola $y=x^2$ in the plane $z=2$ from $(0,0,2)$ to $(1,1,2)$.